Concordance Lecture 8

September 9, 2021 1:35 PM

☐ Mic check ☐ Survey @ end of class
☐ Record ☐ Break after next wk

0 Last Time

Thm (Freedman) If $\Delta_K(t) = 1$, then K is TOP slice.

HW The (positive-untwisted) Whitehead double of a knot has $\Delta_{Wh(K)}(t) = 1$.

Thm $Wh(3_1)$ is not DIFF slice.

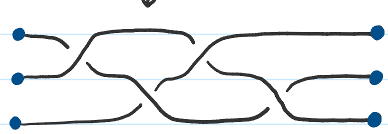
Exotically slice: TOP slice but not DIFF slice ☺ ∃ other "exotic phenomena"

Note Original proof uses Donaldson obstructions (Akbulut)
We will use Slice-Bennequin Inequality (Rudolph, Kronheimer-Mrowka 93)
↳ Hard to prove - will focus on formulation

1 Braids of Artin (1925) Picture Defn

Defn An n-braid is ↓ up to iso rel ∂

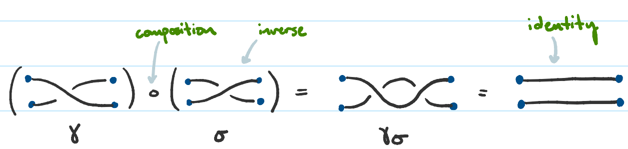
ex $n=3$



non-ex

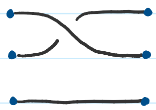
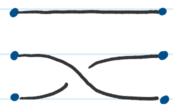
Facts

(A) $B_n = \{n\text{-braids}\}$ forms a group under composition

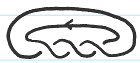


(B) $B_1 = 1$ $\quad B_2 = \mathbb{Z}$ $\quad B_3 = \pi_1(S^3 \setminus 3_1)$

(C) B_n generated by n-1 elements (with positive crossings)

$\sigma_1 =$  $\quad \sigma_2 =$ 

(D) (Alexander's Thm) 1923 Every ort'd knot K has a braid representative β (ie $K=\hat{\beta}$)

ex $\beta =$  $= \sigma_1^3$ $\qquad \hat{\beta} =$  = RHT

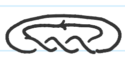
naturally oriented $\qquad$ closure of β

WARNING Not unique

2 Invariants From Braids

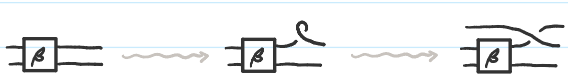
Defn The writhe of an ort'd knot diagram D is $wr(D) := n_+ - n_-$ where $n_\pm$ is the # of pos/neg crossings

pos $\qquad$ neg

Ex $wr($$) = 3$ $\qquad wr($$) = 0$

WARNING Not a knot inv't (in fact, $\forall n \in \mathbb{Z}$ a knot has diagram D with $wr(D) = n$)

- Invariant under R2 and R3, but not R1
- For braids, R1's are "balanced" by # of strands

$\beta_1 =$  $= \beta_2$

β_1 and β_2 have same closure and $wr(\hat{\beta}_1) \pm 1 = wr(\hat{\beta}_2)$

R1's

Thm (Bennequin 82) Let $\beta \in B_n$ and let $\chi_3(\hat{\beta}) = \max\{\chi(S) \mid$ Seif. surf S of $\hat{\beta}\}$. Then

$$\chi_3(\hat{\beta}) \leq n - wr(\beta)$$

☆ Proof uses contact topology

(!) Braids are not unique! Writhe is not an invariant!

↳ It's OK! Different braid reps give better/worse bounds

(!) Can't you make $wr(\beta)$ huge and drive up/down the bound?

↳ No! The $\underbrace{\#\text{ of strands}}_{n}$ balances this

Bennequin Conjectured:

Thm (Slice-Bennequin Inequality) Let $\beta \in B_n$ and let $\chi_4(\hat{\beta}) = \max\{\chi(S) \mid S \subset B^4$ smooth surface, $\partial S = \hat{\beta}\}$. Then

$$\chi_4(\hat{\beta}) \leq n - wr(\beta)$$

☆ Uses Gauge Thy to prove but can understand w/o ☺

Note $\chi_4(\hat{\beta}) \leq 0 \implies \hat{\beta}$ not smoothly slice (sm slice $\iff \chi_4 = 1$)

$\chi(D^2) = 1$

Ex $\beta = \sigma_1^3 =$ 
$\hat{\beta} = 3_1 =$ RHT
$k = 3$
$w = 3$

$\chi_4(3_1) \leq 0$ or $g_4(3_1) \geq 1/2$

Non Ex "Better for positive knots"
$\beta = (\sigma_1^{-1})^3 =$ 
$\hat{\beta} = -3_1 =$ LHT
$k = 3$
$w = -3$

$\chi_4(-3_1) \leq 6$ or $g_4(3_1) \geq -5/2$

(?) What about $Wh(3_1)$ or $Wh(K)$?

3 Strongly Quasipositive Braids + Sliceness (see Rudolph)

Defn A braid β is a ...

(a) positive braid if $\beta = \prod_{i=1}^{m} \sigma_{k_i}$

(b) quasi-positive if $\beta = \prod_{i=1}^{m} w_i \sigma_{k_i} w_i^{-1}$, w_i any braid word in B_n

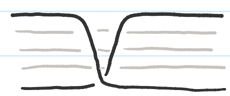
(c) strongly quasi-positive if $\beta = \prod w_i \sigma_{k_i} w_i^{-1}$,

$w_i = \sigma_{k_i - l} \cdots \sigma_{k_i - 2} \sigma_{k_i - 1}$

Boo ALGEBRA! To see these, draw the $w\sigma w^{-1}$ (called band factors)

(b) 

$w\sigma w^{-1}$

(c) 

$w\sigma w^{-1}$

FACT 1. Pos Braids $\subseteq$ Pos Knots $\subseteq$ SQP Braids $\subseteq$ QP Braids (Rudolph)

2. 4_1 not QP

Thm 1 For a QP n-braid $\beta = \prod_{i=1}^{k} w_i \sigma_{k_i} w_i^{-1}$, $\chi_4(\hat\beta) = n - wr(\beta) = n - k$

↳ follows from a deep result of Kronheimer-Mrowka

Thm 2 If β is SQP, then $\chi_4(\hat\beta) = \chi_3(\hat\beta)$

Cor If β is nontriv. SQP, then $\hat\beta$ is not smoothly slice.

Proof $\chi_4(\hat\beta) = \chi_3(\hat\beta) < 1$ because $\hat\beta$ nontrivial $\square$

Thm 3 If K is SQP, then $Wh(K)$ is SQP

Cor $Wh(3_1)$ is not smoothly slice

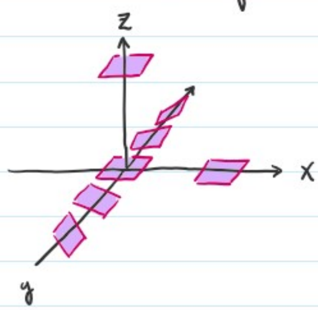
Proof $3_1 = \hat\beta$ for $\beta = \sigma_1^3$, a positive 2-braid

$\therefore$ $Wh(3_1)$ is TOP slice but not SMO slice

4 Legendrian Knots

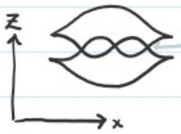
Defn The standard contact structure ξ is a 2D plane field given by $\ker(\alpha)$ where $\alpha = dz - y\,dx$

on $\mathbb{R}^3$



Defn A Legendrian knot Λ is a knot in $\mathbb{R}^3$ that is everywhere tangent to the contact str ξ.

View with front diagram in xz-plane:



y-coordinate given by slope: $0 = dz - y\,dx \Rightarrow y = \frac{dz}{dx}$

Consider up to Legendrian isotopy (iso thru Leg knots)

FACT Every smooth knot has ∞-many Legendrian representatives

Classical Invariants

① Thurston-Bennequin # $\quad tb(\Lambda) = wr(\Lambda) - \#\text{right cusps}$

② Rotation # $\quad r(\Lambda) = \#\text{up cusps} - \#\text{down cusps}$

oriented

Thm (SBI, Rudolph) $\quad \chi_4(\Lambda) \leq -\big(tb(\Lambda) + |r(\Lambda)|\big)$

LATER might use Khovanov homology to prove these SBI's